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CONSISTENCY OF TESTS IN NONCOMMUTATIVE STATISTICS

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Abstract. The paper investigates noncommutative sequential tests. There are defined consistency and uniform consistency of such tests, and then sufficient and necessary conditions for the test to be consistent or uniformly consistent are given.

Holevo in [2] has introduced the notion of noncommutative test as a positive operator X between 0 and 1.

The aim of this paper is to give conditions of consistency and uniform consistency for noncommutative sequential tests.

Throughout the paper let $\mathscr{A}$ be a W^*-algebra, $\{\mathscr{A}_i, i = 1, 2, \ldots\}$ – a sequence of W^*-subalgebras of $\mathscr{A}$ such that $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \ldots$ and $\bigcup_{i=1}^{\infty} \mathscr{A}_i$ generates $\mathscr{A}$.

Definition 1. A *noncommutatitve sequential test* is a sequence of positive operators X_n, $X_n \in \mathscr{A}_n$, $0 \leqslant X_n \leqslant 1$.

Assume that we have two families of normal states on $\mathscr{A}$, say $\{\mu_\Theta\}_{\vartheta\in\Theta}$ and $\{\nu_\lambda\}_{\lambda\in\Lambda}$, where Θ and Λ are arbitrary sets of parameters.

Definition 2. The noncommutative sequential test $\{X_n\}$ will be called *consistent* if, for all $\vartheta \in \Theta$, $\mu_\Theta(X_n)$ converges to 0 and, for all $\lambda \in \Lambda$, $\nu_\lambda(X_n)$ converges to 1.

In the classical theory, the necessary and sufficient conditions for consistency were studied by Kraft [4] who made use of Kakutani's "distance" and "inner product" of probability measures. The noncommutative analogues of these notions are Bures' "distance" and "inner product" of states [1].

Recall that if μ and ν are states on a W^*-algebra $\mathscr{A}$, Bures' "distance" and "inner product" between μ and ν are defined as $d(\mu, \nu) = \inf\|\xi_\mu - \xi_\nu\|$ and $\varrho(\mu, \nu) = \sup|(\xi_\mu, \xi_\nu)|$, respectively, when the infimum and supremum are taken over all vectors ξ_μ, ξ_ν satisfying $\mu = \omega_{\xi_\mu}$, $\nu = \omega_{\xi_\nu}$ in any representation of $\mathscr{A}$ as the algebra of operators acting in some Hilbert space H. d and ϱ

satisfy the formula $d(\mu, \nu)^2 = 2-2\varrho(\mu, \nu)$ and the inequality $d(\mu, \nu)^2 \leqslant \|\mu-\nu\| \leqslant 2d(\mu, \nu)$.

Assume now that Θ and Λ are the sets of parameters with σ-fields of subsets (Θ) and (Λ), respectively. Let p and q be probability measures on $(\Theta, (\Theta))$ and $(\Lambda, (\Lambda))$, respectively. Suppose that $\mu_\vartheta(X)$ and $\nu_\lambda(X)$ are measurable functions of ϑ and λ for each $X \in \mathscr{A}$. Then

$$\mu(X) = \int_\Theta \mu_\vartheta(X)\, p(d\vartheta) \quad \text{and} \quad \nu(X) = \int_\Lambda \nu_\lambda(X)\, q(d\lambda)$$

are normed normal states on $\mathscr{A}$.

Proposition 1. *There exists a consistent sequential test $\{X_n\}$ for p-almost all μ_ϑ against q-almost all ν_λ if and only if μ is orthogonal to ν, i.e. if, for some projector $e \in \mathscr{A}$, $\mu(e) = 1$ and $\nu(e) = 0$.*

Proof. Suppose that $\mu \perp \nu$. Making use of the theorem of Kosaki [3], the lemma of Bures [1] and the methods of Kraft [4], we can easily show that there exists a consistent sequential test for μ against ν. Hence $\int_\Theta \mu_\vartheta(X_n)\, p(d\vartheta)$ converges to 0 and $\int_\Lambda \nu_\lambda(X_n)\, q(d\lambda)$ converges to 1. Thus, for some subsequence $\{X_{n_k}\}$, $\mu_\vartheta(X_{n_k})$ and $\nu_\lambda(X_{n_k})$ converge almost everywhere to 0 and to 1, respectively. Hence we get the desired test.

On the other hand, if such a test $\{X_n\}$ exists, we can find some subnet $\{X_\alpha\}$ ultraweakly convergent to some X between 0 and 1, for which $\mu_\vartheta(X_\alpha)$ tends to 0 and $\nu_\lambda(X_\alpha)$ tends to 1 for almost all ϑ and λ. Hence $\mu(X) = 0$ and $\nu(X) = 1$ and, for some projector e (the spectral projector of X corresponding to the one-point set $\{0\}$, $\mu(e) = 1$ and $\nu(e) = 0$, which completes the proof.

Corollary. *There exists a consistent test between two countable families of states if and only if there exists a consistent test between any pair of states chosen one from each family.*

Definition 3. A sequential test X_n is called *uniformly consistent* if the convergences of $\mu_\vartheta(X_n)$ to 1 and $\nu_\lambda(X_n)$ to 0 are uniform on Λ and Θ, respectively.

As in the classical situation, the existence of uniformly consistent tests can be characterized in terms of the distance in $\mathscr{A}_*$ between the convex hulls of states belonging to each hypothesis. Denote by $C(\{\mu_\vartheta\})$ and $C(\{\nu_\lambda\})$ the convex hulls of the sets $\{\mu_\vartheta\}$ and $\{\nu_\lambda\}$.

We now prove the noncommutative version of the lemma due to LeCam and published in [4]:

Lemma. *The existence of a test X, such that*

$$\text{(i)} \qquad \inf_\vartheta \mu_\vartheta(X) \geqslant \frac{\varepsilon}{2} + \sup_\lambda \nu_\lambda(X),$$

is necessary and sufficient for the distance in $\mathscr{A}_*$ between $C(\{\mu_\vartheta\})$ and $C(\{\nu_\lambda\})$ to be at least ε.

Proof. Consider the real Banach space $\mathscr{A}_*^h$ of all hermitian states on $\mathscr{A}$. It is well-known that the adjoint space to $\mathscr{A}_*^h$ is $\mathscr{A}^h$, i.e. the real space of self-adjoint operators of $\mathscr{A}$.

Assume that inequality (i) holds for some ε and some positive X belonging to the unit sphere S in $\mathscr{A}^h$. This inequality is equivalent to $\varphi(X) \geqslant \varepsilon/2$ for each φ belonging to the set $C = C(\{\mu_\vartheta\}) - C(\{\nu_\lambda\})$. Denote by y the element of S of the form $2X-1$. Evidently, $\varphi(y) \geqslant \varepsilon$, because $\varphi(1) = 0$. We know that $|\varphi(y)| \leqslant \|\varphi\| \cdot \|y\|$, where the norms are from $\mathscr{A}_*$ and $\mathscr{A}$, respectively. Since $\|y\| \leqslant 1$, we have $\|\varphi\| \geqslant \varepsilon$ for each $\varphi \in C$. Thus the distance between $C(\{\mu_\vartheta\})$ and $C(\{\nu_\lambda\})$, equal to $\inf_{\varphi \in C} \|\varphi\|$, is at least ε.

Suppose now that, for each $\varphi \in C$, the inequality $\|\varphi\| > \varepsilon$ holds. Denote by V the open sphere of radius ε and the center at zero in $\mathscr{A}_*^h$. Consider the set $K \subset \mathscr{A}_*^h$ of the form $K = C + V$. It is easy to see that K is an open convex set in $\mathscr{A}_*^h$ which does not contain zero. By the corollary of the Hahn–Banach theorem, there exists a $y \in \mathscr{A}^h$ such that $\psi(y) > 0$ for each ψ belonging to K. Evidently, we may assume that $\|y\| = 1$.

So, we can write $(\varphi+\alpha)(y) > 0$, where $\varphi \in C$ and $\alpha \in V$, or, which is equivalent (because $C+V = C-V$), $\varphi(y) > \alpha(y)$ for all $\varphi \in C$ and all $\alpha \in V$. Hence

$$\inf_{\varphi \in C} \varphi(y) \geqslant \sup_{\alpha \in V} \alpha(y) = \varepsilon \|y\| = \varepsilon.$$

Let $X = \frac{1}{2}y + \frac{1}{2}1$. X is a test and $\inf_{\varphi \in C} \varphi(X) \geqslant \varepsilon/2$, which completes the proof.

Now we are in a position to prove the following

PROPOSITION 2. *There exists a uniformly consistent sequential test for $\{\mu_\vartheta\}$ against $\{\nu_\lambda\}$ if and only if* $\sup \varrho_n(\mu, \nu)$ *converges to* 0 *($\varrho_n(\mu, \nu)$ means here $\varrho(\mu|\mathscr{A}_n, \nu|\mathscr{A}_n)$). The supremum is taken over all μ belonging to $C(\{\mu_\vartheta\})$ and all ν belonging to $C(\{\nu_\lambda\})$.*

Proof. If $\varrho_n(\mu, \nu)$ converges to 0 uniformly, then the distance between the restrictions $\mu|\mathscr{A}_n$ and $\nu|\mathscr{A}_n$ of states μ and ν to subalgebras $\mathscr{A}_n$ converges uniformly to 2. Hence, for some sequence $\{\varepsilon_n\}$ of positive numbers, converging to 0,

$$\inf \|\mu|\mathscr{A}_n - \nu|\mathscr{A}_n\| \geqslant 2 - 2\varepsilon_n.$$

By Lemma, there exists a sequential test $\{X_n\}$ such that

(ii) $$\inf_\vartheta \mu_\vartheta(X_n) \geqslant 1 - \varepsilon_n + \sup_\lambda \nu_\lambda(X_n).$$

Hence $\inf_{\vartheta} \mu_{\vartheta}(X_n)$ converges to 1 and $\sup_{\lambda} \nu_{\lambda}(X_n)$ converges to 0.

On the other hand, if $\inf_{\vartheta} \mu_{\vartheta}(X_n) \geqslant 1-\varepsilon_n/2$ and $\sup_{\lambda} \nu_{\lambda}(X_n) \leqslant \varepsilon_n/2$, then (ii) holds.

The simple use of Lemma gives the proof.

The most important example of the W^*-algebra $\mathscr{A}$ with the sequence of subalgebras $\mathscr{A}_n$ generating $\mathscr{A}$ is the infinite tensor product of W^*-algebras. In this case we can define product states and – asymptotically – product states and consider the existence tests for such states.

Similarly as the consistent test, we can also define and consider consistent estimates. The reader will be able to find these considerations in our next papers.

Remark. Theorems in the paper are the noncommutative analogues of the corresponding theorems of [4]. The proofs are based on similar ideas with the use of notions and facts characteristic for the theory of W^*-algebras.

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